A LARGE DEVIATION PROPERTY VIA THE RENEWAL THEOREM

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Abstract. In this note we prove a large deviation property for the occupation time of a point by a continuous time Markov chain. The main step is to prove that the Laplace exponent satisfy a renewal equation.

Let $(X_t, \mathcal{F}_t, t \geq 0; \mathbb{P}_x, x \in \chi)$ be a continuous time Markov process with values in a discrete state space χ , with $0 \in \chi$ a distinguished point ; we let $\mathbb{P} = \mathbb{P}_0$ and $\mathbb{E}\left[\right]$ denote expectation under $\mathbb{P}_0$.. Let $r(t) = \mathbb{P}(X_t = 0)$ and ϕ be its Laplace transform

(1) $$\alpha \in \mathbb{R} \to \phi(\alpha) = \int_0^\infty e^{\alpha t} r(t)\, dt \ \in [0, +\infty] .$$

We shall make tho following technical assumption : if $\phi(\alpha) < +\infty$ then the function $t \to e^{\alpha t} r(t)$ is directly Riemann integrable on $(0, +\infty)$ (see Feller[2], chap XI). Observe that this is satisfied if $t \to r(t)$ is decreasing.

Eventually let us denote by a the convergence abcissa of the Laplace transform:

$$a = \sup \{\alpha : \phi(\alpha) < +\infty\} .$$

Since $0 \leq r(t) \leq 1$ the function ϕ is increasing, finite on $(-\infty, 0)$, ie $a \geq 0$ (it is furthermore C^∞ on $(-\infty, a)$). We can now state the main result of this note

Theorem 1. *For $\lambda \geq 0$, we define :*

$$u(t) = \mathbb{E}\left[e^{\lambda \int_0^t 1_{(X_s=0)}\, ds}\right] .$$

Then $\frac{1}{t} \log u(t)$ converges as $t \to \infty$ to the function

(2) $$\Lambda(\lambda) = \begin{cases} 0 & \text{if } \lambda \leq \frac{1}{\phi(0)} \\ -\alpha(\lambda) & \text{if } \lambda > \frac{1}{\phi(0)} \end{cases} ,$$

where $\alpha(\lambda)$ is for $\lambda > \frac{1}{\phi(0)}$ the unique solution of $\lambda\phi(\alpha) = 1$ (and where we have the notation $\frac{1}{+\infty} = 0$.

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From the implicit function theorem we deduce that α is smooth on $(\frac{1}{\phi(a)}, +\infty)$, and thus that the legendre transform Λ^* has for set of exposed points $\mathcal{F} \supset (0, +\infty)$. Hence we can apply Gärtner–Ellis theorem (see e.g. Dembo and Zeitouni's book[1]) to get the

Corollary 2. *The family $(\mu_t)_{t\geq 0}$ of distributions of $\frac{1}{t}\int_0^t 1_{(X_s=0)}\,ds$ under $\mathbb{P}_0$ satisfy a large deviations principle with good rate function:*

$$\Lambda^*(x) = \sup_\lambda(\lambda x - \Lambda(\lambda))\,.$$

Proof of the Theorem. Observe first that

$$\begin{aligned} u(t) &= \mathbb{E}\left[e^{\lambda\int_0^t 1_{(X_s=0)}\,ds}\right] \\ &= \mathbb{E}\left[1 + \lambda\int_0^t 1_{(X_s=0)} e^{\int_0^s 1_{(X_s=u)}\,du}\,ds\right] \\ &= 1 + \lambda\int_0^t \mathbb{E}\left[1_{(X_s=0)} e^{\int_0^s 1_{(X_s=u)}\,du}\right] ds \\ &= 1 + \int_0^t v(s)\,ds\,. \end{aligned}$$

Then, recall that $r(t) = \mathbb{P}_t\phi(0)$ whith $\phi(x) = 1_{(x=0)}$, where P_t denotes the semi group of X,

$$\begin{aligned} v(t) &= \mathbb{E}\left[\lambda 1_{(X_t=0)}(1 + \int_0^t \lambda 1_{(X_s=0)} e^{\int_0^s 1_{(X_s=u)}\,du})\right] \\ &= \lambda r(t) + \lambda^2\int_0^t \mathbb{E}\left[1_{(X_t=0)} 1_{(X_s=0)} e^{\int_0^s 1_{(X_s=u)}\,du})\right] ds \\ &= \lambda r(t) + \lambda^2\int_0^t \mathbb{E}\left[P_{t-s}\phi(X_s) 1_{(X_s=0)} e^{\int_0^s 1_{(X_s=u)}\,du})\right] ds \\ &= \lambda r(t) + \lambda\int_0^t r(t-s)v(s)\,ds\,. \end{aligned}$$

Assume first that $\lambda\phi(a) > 1$. Then, there exists $\alpha < a$ such that $\lambda\phi(\alpha) = 1$. We let $w(t) = e^{\lambda t}v(t)$ and $\rho(t) = \lambda e^{\alpha t} r(t)$. The ρ is a probability density, and w is a solution, in fact the unique solution, of the renewal equation

$$w = \rho + \rho * w\,.$$

Since w is locally bounded, and by assumption ρ is directly Riemann integrable, we obtain by the renewal theorem that

$$\lim_{t\to+\infty} w(t) = (\int_0^\infty s\rho(s)\,ds)^{-1}\int_0^\infty \rho(s)\,ds = \frac{\phi(\alpha)}{\phi'(\alpha)}\,. \tag{3}$$

This yields immediately that if $\alpha(\lambda) < 0$, then

$$\lim_{t\to+\infty}\frac{1}{t}\log u(t) = \lim_{t\to+\infty}\frac{1}{t}\log v(t) = -\alpha(\lambda)$$

and if $\alpha(\lambda) \geq 0$, then $\lim_{t\to+\infty} \frac{1}{t} \log u(t) = 0$.

Assume now that $0 < \lambda < \frac{1}{\phi(0)}$. Then $\phi(0) < +\infty$, and $A_\infty = \int_0^\infty 1_{(X_s=0)}\, ds$ is under $\mathbb{P}$ distributed as an exponential random variable of expectation $\mathbb{E}\left[A_\infty\right] = \phi(0)$. Indeed, Markov property shows that A_∞ has no memory: let $A_t = \int_0^t 1_{(X_s=0)}\, ds$ and $\tau_u = \inf\left\{t > 0 : A_t > u\right\}$. Then, by the strong Markov property,

$$\begin{aligned}\mathbb{P}\left(A_\infty > t + s \mid A_\infty > s\right) &= \mathbb{P}\left(s + A_\infty \circ \theta_{\tau_s} > t \mid \tau_s < \infty\right) \\ &= \mathbb{P}_{X_{\tau_s}}\left(A_\infty > t\right) = \mathbb{P}\left(A_\infty > t\right).\end{aligned}$$

Therefore, $\mathbb{E}\left[e^{\lambda A_\infty}\right] < +\infty$ and $u(t) \to u(\infty) < +\infty$. □

References

[1] A. Dembo and O. Zeitouni, *Large deviations techniques and applications*, vol. 38 of Applications of Mathematics (New York), Springer-Verlag, New York, second ed., 1998.

[2] W. Feller, *An introduction to probability theory and its applications. Vol. I.*, John Wiley & Sons, 1950.

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