Exponential Functionals of Lévy Processes

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ABSTRACT. The distribution of the terminal value A_∞ of the exponential functional

$$A_t(\xi) = \int_0^t e^{\xi_s}\, ds$$

of a Lévy process $(\xi_t)_{t\geq 0}$ plays an important role in Mathematical Physics and Mathematical Finance. We show how this distribution can be computed by means of Lamperti's transformation and generalized Ornstein–Uhlenbeck processes.

1 Introduction

The exponential functional of a Lévy process $(\xi_t\, ;\, t \geq 0)$ is the process

$$A_t(\xi) = \int_0^t e^{\xi_s}\, ds, \qquad t \geq 0.$$

During the 1990s, the study of distributional properties of exponential functionals played an important role in the two domains of mathematical finance and mathematical physics. The related papers dealing with mathematical finance explore essentially three directions:

- Pricing of Asian options [14, 44].
- Determining the law of a perpetuity [3, 12, 43, 35].
- Risk theory [34, 30, 18, 17, 31].

In mathematical physics, the exponential functional is a key quantity to study a one-dimensional diffusion in a random Lévy environment [21, 20, 29, 9, 8, 4]. It also appears for instance in random dynamical systems (see, e.g., [1, p. 99]).

In this paper, we try to give a unified framework with which we can interpret the results of our two previous papers [5, 6]. We focus attention on the distributional properties of the terminal value A_∞. This is sufficient, at least theoretically, since killing ξ at an independent exponential time S_θ of parameter θ yields, through a Laplace transform inversion, the law at fixed time t:

$$\mathbb{E}\left[f(A_{S_\theta})\right] = \int_0^\infty \theta\, e^{-\theta t}\, \mathbb{E}\left[f(A_t)\right] dt.$$

Throughout this note, a selected choice of examples illustrates our two main tools: Lamperti's transformation and generalized Ornstein–Uhlenbeck processes.

The paper is organized as follows. After this introduction, Section 2 contains a brief summary of the properties of Lamperti's transformation. In essence the space and time change

$$e^{\xi_t} = X_{A_t} \qquad (t \geq 0),$$

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establishes a one-to-one correspondence between one-dimensional Lévy processes and semistable Markov processes (Markov processes on $(0, +\infty)$ which are 1-self-similar). Through this correspondence, we obtain, if $\xi_t \to -\infty$ almost surely, the identity

$$A_\infty(\xi) = T_0(X) = \inf\{u > 0 : X_u = 0 \text{ or } X_{u-} = 0\},$$

which proves to be a powerful way to determine the law of A_∞.

Section 3 is devoted to the study of the consequences of another transformation. Given (ξ, η) a two-dimensional Lévy process, the formula

$$X_t^x = e^{\xi_t}\left(x + \int_0^t e^{-\xi_{s-}}\, d\eta_s\right) \qquad (t \geq 0, x \in \mathbb{R})$$

defines a homogeneous Markov process which we call the generalized Ornstein–Uhlenbeck process associated to the pair (ξ, η). Under some mild assumptions, we show that the law of the variable $A_\infty(\xi, \eta) = \int_0^\infty e^{\xi_{s-}}\, d\eta_s$, if it exists, is the unique invariant probability law of this generalized Ornstein–Uhlenbeck process. This important remark has also been made by a number of authors (see, e.g., [38, 41]), sometimes in a disguised manner, e.g, the technique found in the works by Paulsen et al. [34, 35, 30] is very closely related to this method.

Eventually, Section 4 presents our attempt to solve a two-dimensional problem, namely to determine the joint law of

$$\left(\int_0^\infty e^{2(B_s - \nu_1 s)}\, ds,\ \int_0^\infty e^{2(B_s - \nu_2 s)}\, ds\right) \qquad (\nu_1 > 0, \nu_2 > 0).$$

2 The Lamperti transformation

Given a one-dimensional Lévy process $(\xi_t\,;\, t \geq 0)$, we may define another Markov process X via the time change $t \mapsto A_t(\xi)$, namely

$$e^{\xi_t} = X_{A_t}, \qquad A_t = \int_0^t e^{\xi_s}\, ds. \tag{2.1}$$

Introducing the continuous inverse $u \mapsto C_u$ of A, we may write alternatively

$$X_u = e^{\xi_{C_u}}, \qquad C_u = \int_0^u \frac{ds}{X_s} \quad (u < A_\infty). \tag{2.2}$$

From the relation (2.1), we easily derive the following identity.

Proposition 2.1. *Suppose that almost surely $\xi_t \to -\infty$. Then the exponential functional, taken at time $t = +\infty$, is the first hitting time of level 0 by the Markov process X defined by the Lamperti relation* (2.1). *More precisely,*

$$A_\infty(\xi) = T_0(X) = \inf\{t > 0 : X_t = 0 \text{ or } X_{t-} = 0\}. \tag{2.3}$$

From the independence and stationarity of the increments of ξ, we may deduce the scaling property of X: for every $a > 0$,

$$\left(\frac{1}{a} X_{at}, 0 \leq t < T_0\,;\, \mathbb{P}_{ax}\right) \stackrel{d}{=} (X_t, 0 \leq t < T_0\,;\, \mathbb{P}_x), \tag{2.4}$$

where $\mathbb{P}_x$ denotes the law of X starting from $x > 0$ (hence ξ starts from $\log x$). Lamperti calls semistable Markov process a strong Markov process on the state space $(0, +\infty)$ that satisfies this scaling property. One of the deep results of Lamperti [25] is the existence of a one-to-one correspondence between Lévy processes on the line and semistable Markov processes.

It is quite straightforward to derive the following relation between the infinitesimal generators (the correspondence between the domains being obvious)

$$L^X f(x) = \frac{1}{x} L^\xi (f \circ \exp)(\log x), \qquad L^\xi g(\xi) = e^\xi L^X (g \circ \log)(e^\xi). \tag{2.5}$$

In particular, if we define the Lévy–Laplace exponent of ξ by

$$\mathbb{E}\left[e^{m\xi_t}\right] = e^{t\psi(m)} \qquad (m \in \mathbb{R}, t \geq 0), \tag{2.6}$$

then we have, if $\psi(m) < +\infty$,

$$L^X(f_m) = \psi(m) f_{m-1}, \qquad \text{where} \quad f_m(x) = x^m. \tag{2.7}$$

Indeed, if $g_m(\xi) = f_m(e^\xi) = e^{m\xi}$, then we have $P_t^\xi g_m = e^{t\psi(m)} g_m$, and thus $L^\xi g_m = \psi(m) g_m$, from which we deduce (2.7).

2.1 *Example 1: Dufresne's perpetuity distribution*

We take $\xi_t = 2(B_t - \nu t)$, where B is a standard Brownian motion, and $\nu > 0$.

Proposition 2.2 ([12], [44], [38, Proposition 3]). *We have*

$$\int_0^\infty e^{2(B_s - \nu s)}\, ds \overset{d}{=} \frac{1}{2\gamma_\nu},$$

where γ_ν is a gamma random variable with parameter $\nu > 0$.

Proof. From the generator of ξ,

$$L^\xi g(\xi) = 2g''(\xi) - 2\nu g'(\xi),$$

we deduce the generator of X,

$$L^X f(x) = 2xf''(x) + 2(1-\nu) f'(x).$$

Hence, X is a squared Bessel process, starting from $X_0 = 1$, of dimension $\delta = 2(1-\nu)$, killed when it first hits 0. It is a standard fact of Bessel process studies [19, 22, 16, 36] that the first hitting time of 0 is $T_0(X) \overset{d}{=} \frac{1}{2\gamma_\nu}$ and this concludes our proof. □

2.2 *Example 2: The Cauchy process*

We suppose here that $X_t = |C_t|$ is the absolute value of a Cauchy process $(C_t, t \geq 0)$.

Proposition 2.3. *The Lévy process associated by Lamperti's relation with the absolute value of a Cauchy process has infinitesimal generator*

$$L^\xi g(\xi) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\cosh \eta}{(\sinh \eta)^2} (g(\xi + \eta) - g(\xi) - \eta\, g'(\xi) 1_{(|\eta| \leq 1)})\, d\eta.$$

Then

$$\forall t \geq 0,\ \forall \lambda \in \mathbb{R},\ \mathbb{E}\left[\exp(i\lambda\xi_t)\right] = e^{-t\lambda \tanh(\frac{\pi\lambda}{2})}, \tag{2.8}$$

so that

$$(\xi_t; t \geq 0) \stackrel{d}{=} \left(\int_0^{\tau_t} 1_{(|B_s| \leq \pi/2)}\, d\gamma_s; t \geq 0\right), \tag{2.9}$$

where B and γ are two independent standard Brownian motions, and τ_t is the inverse of the local time at 0 *of B.*

Proof. From relation (2.5), we easily check that ξ is clearly the Lévy process corresponding to X, since we have

$$L^X f(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{dy}{y^2}(f(|x+y|) - f(x) - y1_{(|y|\leq a)} f'(x))$$

(this expression does not depend on a, we choose $a = x$ and we perform a change of variables). Since the Cauchy process never hits 0 (this goes back at least to Spitzer (1958); see also [37] for some asymptotic results), we have $A_\infty = +\infty$ almost surely. On the other hand, it is well known (see, e.g., Knight [23, Section 2]), that the Laplace transform in $\frac{\lambda^2}{2}$ of $\int_0^{\tau_t} 1_{(|B_s|\leq\pi/2)}\, ds$ is given by the right hand side of (2.8). The identity (2.9) follows, since the right member is also a Lévy process. □

Remark 2.1. Analogous results ([7], in preparation) may be obtained when X is the norm of a multidimensional Cauchy process, which may be defined as a subordinated Bessel process.

3 The generalized Ornstein–Uhlenbeck process

Given a two-dimensional Lévy process (ξ, η) starting from $(0, 0)$, the process

$$X_t^x = e^{\xi_t}\left(x + \int_0^t e^{-\xi_{s-}}\, d\eta_s\right) \qquad (t \geq 0, x \in \mathbb{R}), \tag{3.1}$$

is a homogeneous Markov process called the generalized Ornstein–Uhlenbeck process driven by (ξ, η). Indeed, given $t \geq 0$, we only need to observe that the Lévy process

$$(\bar{\xi}_s = \xi_{t+s} - \xi_t, \bar{\eta}_s = \eta_{t+s} - \eta_t; s \geq 0)$$

is independent of $\mathcal{F}_t = \sigma(\xi_u, \eta_u, u \leq t)$ and distributed as (ξ, η). The Markov property is now an immediate consequence of the relation

$$X_{t+s}^x = e^{\bar{\xi}_s}\left(X_t^x + \int_0^s e^{-\bar{\xi}_{u-}}\, d\bar{\eta}_u\right).$$

The deep relationship between exponential functionals and Lévy processes is illustrated by the following.

Theorem 3.1. *If ξ and η are independent Lévy processes (in particular, if $\eta_t = t$), then for each fixed time $t > 0$ we have the identity in distribution*

$$X_t^x \stackrel{d}{=} e^{\xi_t} x + \int_0^t e^{\xi_{s-}}\, d\eta_s.$$

Furthermore, if

1. $\xi_t \to -\infty$ *almost surely;*
2. $A_\infty = A_\infty(\xi, \eta) = \int_0^\infty e^{\xi_{s-}}\, d\eta_s$ *is defined and almost surely finite;*

then, the law of A_∞ *is the unique invariant probability measure of the generalized Ornstein–Uhlenbeck process* X *driven by* (ξ, η).

The key ingredient of the proof is the elementary

Lemma 3.1 (time reversal property of Lévy processes). *Let* Z *be a* d*-dimensional Lévy process. Then, for fixed time* $t > 0$,

$$(\tilde{Z}_s = Z_t - Z_{(t-s)-}, 0 \leq s \leq t)$$

is a d*-dimensional Lévy process which has the same distribution as* $(Z_s, 0 \leq s \leq t)$.

Proof of Theorem 3.1. Using first the independence of ξ and η, and then Lemma 3.1, we obtain

$$\begin{aligned} X_t &= xe^{\xi_t} + e^{\xi_t}\int_0^t e^{-\xi_{s-}}\, d\eta_s = xe^{\xi_t} + \int_0^t e^{\xi_t - \xi_{s-}}\, d\eta_s \\ &= xe^{\tilde{\xi}_t} + \int_0^t e^{\tilde{\xi}_{u-}}\, d\tilde{\eta}_u \overset{d}{=} xe^{\xi_t} + \int_0^t e^{\xi_{u-}}\, d\eta_u. \end{aligned}$$

Since $\xi_t \to -\infty$ almost surely, we get that for every starting point x, $X_t \xrightarrow[t\to\infty]{d} A_\infty$, and this yields the desired result since X is a Markov process. □

3.1 *Example 3: The deterministic case* $\eta_t = t$

The infinitesimal generators L^ξ and L^X, respectively of the process ξ and of the associated generalized Ornstein–Uhlenbeck process X, are linked as follows (cf. [6, p. 80]):

$$L^X f(x) = f'(x) + L^\xi (f \circ \exp)(\log x). \tag{3.2}$$

The case $\xi_t = 2(B_t - \nu t)$

As a consequence of Theorem 3.1, the density ρ of the law of $\int_0^\infty e^{2(B_s - \nu s)}\, ds$ is the solution of

$$(x^2 \rho)'' - \left(\left(\frac{1}{2} + x(1-\nu) \right) \rho \right)' = 0, \tag{3.3}$$

from which we easily recover Dufresne's perpetuity distribution.

The case $\xi_t = \alpha t + \varepsilon S_{N_t}$

Suppose that $S_{N_t} = \sum_{i=1}^{N_t} Y_i$ is a compound Poisson process, that is, $(N_t)_{t\geq 0}$ is a Poisson process with intensity $\beta > 0$, independent of the i.i.d. sequence $(Y_i)_{i\geq 1}$ with common distribution ν. Let us consider the process $\xi_t = \alpha t + \varepsilon S_{N_t}$, where ε is either 1 or -1 (ε is not a random variable!). The infinitesimal generator of the associated generalized Ornstein–Uhlenbeck process is

$$L^X f(x) = (1 + \alpha x) f'(x) + \beta \int (f(xe^{\varepsilon y}) - f(x))\, \nu(dx). \tag{3.4}$$

In the particular case of exponential jumps, $\nu(dx) = \mu e^{-\mu x} 1_{(x>0)}\, dx$, $A_\infty(\xi) = \int_0^\infty e^{\xi_s}\, ds$ is almost surely finite as soon as $\mathbb{E}[\xi_1] = \alpha + \frac{\varepsilon\beta}{\mu} < 0$, and its density ρ is the solution of

$$-((1+\alpha x)\rho)' + \beta\,\mu \int_0^\infty e^{-(\mu+\varepsilon)y}\, \rho(xe^{-\varepsilon y})\, dy = \beta\,\rho(x). \tag{3.5}$$

We then recover the laws of the variables $A_\infty(\xi)$ we had obtained in [5]: in Proposition 5.2 of [5], if γ_a and $\beta(a,b)$ are gamma and beta distributed variables, one should read the following identity: $A_{T_\lambda} \stackrel{d}{=} \gamma_{\alpha+\beta} \frac{\beta(1,\alpha+m-\lambda-1)}{\gamma_m}$, so that $A_\infty \stackrel{d}{=} \gamma_{\alpha+\beta}/\gamma_{1-\alpha}$ if $\alpha+\beta \geq 1$ and $\alpha < 1$, and $A_\infty = +\infty$ almost surely otherwise.

Remark 3.1. One may also replace η by an α-stable process independent from ξ. The results obtained in that case are immediate consequences of those given when $\eta_t = t$ (cf. [6]).

3.2 *Example 4: The risk model*

We shall use the economic model defined by Paulsen [34]: the surplus generating process P, the inflation generating process I, and the return on investment process R are assumed to be such that (P, I, R) is a three-dimensional Lévy process. In particular, they are semimartingales. Under reasonable assumptions on the jump processes, Paulsen showed that the risk process is a generalized Ornstein–Uhlenbeck process

$$X_t^x = e^{\xi_t}\left(x + \int_0^t e^{-\xi_{s-}}\, d\eta_s\right), \tag{3.6}$$

where (ξ, η) is a two-dimensional Lévy process starting from zero depending on (I, R, P) and x is the initial wealth.

The ruin probability

The ruin probability is defined for $x > 0$ by

$$R(x) = \mathbb{P}_x(T_0 < +\infty), \quad \text{where} \quad T_0 = \inf\{t > 0 : X_t < 0, \text{ or } X_{t-} < 0\}$$

and where $\mathbb{P}_x$ denotes the law of X starting from x. It is usually determined by the distribution of the exponential functional

$$A_t = A_t(-\xi, \eta) = \int_0^t e^{-\xi_{s-}}\, d\eta_s.$$

Proposition 3.1. *Assume that η is a continuous Lévy process, i.e., a Brownian motion with drift, and that the exponential functional*

$$A_\infty = \int_0^\infty e^{-\xi_{s-}}\, d\eta_s$$

is defined and finite almost surely. Then, if $\mathbb{P}(A_\infty < -x) > 0$, the ruin probability is given by

$$R(x) = \mathbb{P}_x(T_0 < +\infty) = \frac{\mathbb{P}(A_\infty < -x)}{\mathbb{P}(A_\infty < 0)}.$$

Proof. Since $t \mapsto A_t(\omega)$ is a function converging to A_∞ and

$$T_0 = T = \inf\{t > 0 : x + A_t < 0\},$$

on the event $(A_\infty < -x)$ we have $\mathbb{P}_x$ almost surely $T < +\infty$. Furthermore, thanks to the independence and stationarity of increments, the process

$$(\tilde{\xi}_t = \xi_{T+t} - \xi_T, \tilde{\eta}_t = \eta_{T+t} - \eta_T\,; t \geq 0)$$

is a Lévy process independent of $\mathcal{F}_T$ whose conditional law on $(T < +\infty)$ is the same as the law of (ξ, η). Therefore,

$$A_\infty = A_T + e^{-\xi_T}\tilde{A_\infty} = -x + e^{-\xi_T}\tilde{A_\infty} \qquad \text{on } (T < +\infty),$$

where $\tilde{A_\infty}$ is independent of $\mathcal{F}_T$ and distributed as A_∞. Hence, for $x \geq 0$,

$$\begin{aligned}
\mathbb{P}_x(A_\infty < -x) &= \mathbb{P}_x(A_\infty < -x, T < +\infty) \\
&= \mathbb{P}_x\left(e^{-\xi_T}\tilde{A_\infty} < 0 \mid T < +\infty\right)\mathbb{P}_x(T < \infty) \\
&= \mathbb{P}_x\left(\tilde{A_\infty} < 0\right)\mathbb{P}_x(T < +\infty) = \mathbb{P}_x(A_\infty < 0)\mathbb{P}_x(T < +\infty).
\end{aligned}$$

□

The independent case

We suppose that ξ and η are independent continuous Lévy processes, that is, Brownian motions with drift. Thanks to the scaling property of Brownian motion, we may restrict ourselves to

$$\xi_t = B_t - \nu t, \qquad \eta_t = W_t + \mu t,$$

where $\nu > 0$, $\mu \in \mathbb{R}$, and B, W are independent Brownian motions on the line. Paulsen [34] established the following result, which was rediscovered and generalized by Baldi et al. [2].

Proposition 3.2. *The exponential functional*

$$A_\infty(\xi, \eta) = \int_0^\infty e^{B_t - \nu t}(dW_t + \mu dt)$$

follows a Pearson type IV distribution of parameters ν *and* 2μ, *that is, it has for density*

$$f(x) = (\mathit{constant})(1 + x^2)^{-(\nu+\frac{1}{2})}\exp(2\mu \arctan x).$$

We shall prove in Proposition 3.4 below a slightly more general statement.

The dependent case

If we no longer assume ξ and η to be independent, then the situation becomes more complex, as we shall soon see. Suppose that η is a Brownian motion with drift

$$\eta_t = W_t + \mu t, \qquad \xi_t = \sigma W_t + \chi_t,$$

where χ is a Lévy process independent from the Brownian motion W. There the risk process may be seen as

$$X_t^x = \frac{1}{S_t e^{\bar{\chi}_t}}\left(x + \int_0^t e^{\bar{\chi}_s} dS_s\right),$$

where $\bar{\chi}_t = \chi_t - rt$ and $S_t = S_0 e^{\sigma W_t + rt}$ is a geometric Brownian motion, usually a good candidate to model the price of an asset.

We shall use the integration by parts formula for the Skorohod anticipating integral (see, e.g., Nualart's book [32]):

$$F \int u_s \, dW_s = \int F \, u_s \, dW_s + \int D_s F u_s \, ds \qquad (F \in \mathbf{D}^{1,2}, u \in \mathbf{L}^{1,2}). \tag{3.7}$$

Proposition 3.3. *Define $\bar{\eta}_t = W_t + (\sigma + \mu)t = \eta_t + \sigma t$. Assume furthermore that $\xi_t \to -\infty$ almost surely, and that $A_\infty = A_\infty(\xi, \bar{\eta}) = \int_0^\infty e^{\xi_s} d\bar{\eta}_s$ is defined and almost surely finite. Then, the law of A_∞ is the unique invariant probability measure of the generalized Ornstein–Uhlenbeck process X.*

Proof. The integration by parts formula yields

$$\begin{aligned} X_t &= x e^{\xi_t} + \mu e^{\xi_t} \int_0^t e^{-\xi_s} \, ds + e^{\xi_t} \int_0^t e^{-\xi_s} \, dW_s \\ &= x e^{\xi_t} + \mu \int_0^t e^{\xi_t - \xi_s} \, ds + \int_0^t e^{\xi_t - \xi_s} \, dW_s + \int_0^t D_s(e^{\xi_t}) e^{-\xi_s} \, ds. \end{aligned}$$

But, by independence of ξ and W, and the chain rule, we have

$$D_s(e^{\xi_t}) = D_s(e^{\sigma W_t + \chi_t}) = e^{\chi_t} D_s(e^{\sigma W_t}) = e^{\chi_t} \sigma e^{\sigma W_t} D_s(W_t) = \sigma e^{\xi_t} 1_{(s \le t)}.$$

Therefore,

$$X_t = x e^{\xi_t} + (\sigma + \mu) \int_0^t e^{\xi_t - \xi_s} \, ds + \int_0^t e^{\xi_t - \xi_s} \, dW_s.$$

Thanks to Lemma 3.2, we can introduce the time reversed processes

$$X_t = x e^{\tilde{\xi}_t} + (\sigma + \mu) \int_0^t e^{\tilde{\xi}_u} \, du + \int_0^t e^{\tilde{\xi}_u} \, d\tilde{W}_u,$$

and thus

$$X_t \stackrel{d}{=} x e^{\xi_t} + \int_0^t e^{\xi_s} \, d\bar{\eta}_s.$$

We now conclude as in Theorem 3.1. □

The following Lemma is an immediate consequence of the fact that the Skorohod integral generalizes not only the usual Itô integral, but also the so-called backward stochastic integral (see Rosen and Yor [40], Kunita [24], Pardoux and Protter [33]).

Lemma 3.2. *Let $(W_t \, ; t \ge 0)$ be a standard Brownian motion and $(s, x) \mapsto f(s, x)$ a measurable function such that*

$$\mathbb{E}\left[\int_0^t f(s, W_s)^2 \, ds\right] < +\infty.$$

Then, if $(\tilde{W}_u = W_t - W_{t-u}, 0 \le u \le t)$ denotes the time reversed process,

$$\int_0^t f(t - s, W_t - W_s) \, dW_s = \int_0^t f(u, \tilde{W}_u) \, d\tilde{W}_u.$$

Application

We suppose here that the Lévy process χ is also a Brownian motion with drift. Thanks to the scaling property of Brownian motion, we may restrict ourselves to the case

$$\eta_t = W_t + \mu t, \quad \xi_t = \cos(\theta) W_t + \sin(\theta) B_t - \nu t,$$

where B is a Brownian motion independent from W and $\nu > 0$.

Proposition 3.4. *The unique invariant probability law of the generalized Ornstein–Uhlenbeck process X is the law of Z such that $\frac{Z+\cos\theta}{\sin\theta}$ follows a Pearson type IV distribution of parameters ν and $\kappa = 2\frac{(\frac{1}{2}+\nu)\cos\theta+\mu}{\sin\theta}$, i.e., such that $\frac{Z+\cos\theta}{\sin\theta}$ has density*

$$f(x) = (constant)\,(1+x^2)^{-(\frac{1}{2}+\nu)} \exp(\kappa \arctan x).$$

Furthermore, if $\nu > 1$, then the unique invariant probability law of the generalized Ornstein–Uhlenbeck X is the law of

$$A_\infty = A_\infty(\xi, \bar{\eta}) = \int_0^\infty e^{\cos\theta W_s + \sin\theta B_s - \nu s}\, d(W_s + (\mu + \cos(\theta))s).$$

Proof. The second part of the proposition follows immediately from Proposition 3.3, once we have remarked that A_∞ is well defined and finite, since

$$\mathbb{E}\left[\int_0^\infty e^{2\xi_s}\, ds\right] = \frac{1}{2(\nu-1)}, \quad \mathbb{E}\left[\int_0^\infty e^{\xi_s}\, ds\right] = \frac{1}{(\nu-\frac{1}{2})}.$$

We can easily derive from Itô's formula the infinitesimal generator of X:

$$Lf(x) = \frac{1}{2}(1+x^2+2x\cos\theta) f''(x) + \left(\mu + \cos\theta - \left(\nu - \frac{1}{2}\right)x\right) f'(x) \quad (f \in C_b^2).$$

The positive measure ρ is an invariant measure for X if and only if it satisfies

$$\int Lf(x)\,\rho(dx) = 0 \quad (f \in C_b^2).$$

Assuming that ρ has a density h with respect to Lebesgue measure, such that h is smooth enough, we are looking for a solution of the differential equation

$$\frac{1}{2}\frac{d^2}{dx^2}((1+x^2+2x\cos\theta)\,h) - \frac{d}{dx}\left(\left(\mu + \cos\theta - \left(\nu - \frac{1}{2}\right)x\right)h\right) = 0,$$

that is,

$$\frac{1}{2}(1+x^2+2x\cos\theta)\,h''(x) + ((3/2+\nu)x + (\cos\theta - \mu))\,h'(x) + \left(\nu + \frac{1}{2}\right)h(x) = 0.$$

We find it is enough for h to satisfy

$$\frac{1}{2}(1+x^2+2x\cos\theta)\,h'(x) + \left(\left(\nu + \frac{1}{2}\right)x - \mu\right)h(x) = 0$$

and this equation is easily solved, giving a Pearson type IV density. □

Another example

Assume now that $\xi_t = W_t - rt$ is a Brownian motion with negative drift and that $\eta_t = \sum_{i=1}^{N_t} Y_i$ is a compound Poisson process of parameter β and jump distribution $\nu(dx)$ (cf. Example 3). The generalized Ornstein–Uhlenbeck process solves the stochastic differential equation

$$dX_t = X_{t-}(d\xi_t + \frac{1}{2}d\langle\xi,\xi\rangle_t) + d\eta_t = X_{t-}(dW_t + (\frac{1}{2} - r)\,dt) + d\eta_t.$$

Therefore, its infinitesimal generator is

$$L^X f(x) = (\frac{1}{2} - r)xf'(x) + \frac{1}{2}x^2 f''(x) + \beta \int (f(x+y) - f(x))\,\nu(dy).$$

In the particular case of exponential jumps, $\nu(dx) = \mu e^{-\mu x} 1_{(x>0)}\,dx$, we assume that the law of A_∞ is absolutely continuous with density h. From the invariance property

$$\int Lf(x)\,h(x)\,dx = 0 \qquad (f \in C_b^2),$$

we derive the integrodifferential equation

$$(r - \frac{1}{2})(xh)' + \frac{1}{2}(x^2 h)'' + \beta\,\mu \int_0^x e^{-\mu y} h(x-y)\,dy = \beta\,h(x).$$

We may now check Nilsen and Paulsen's result [30]: if $b = r((1 + 2\beta/r^2)^{\frac{1}{2}} - 1)$, then

$$h(x) = \frac{\Gamma(1+2r+b)\mu^b x^{b-1}}{\Gamma(2r)\Gamma(1+b)\Gamma(b)} \int_0^1 y^{2r+b-1}(1-y)^b e^{-\mu xy}\,dy.$$

In other words, A_∞ is distributed as X/Y, where $X \sim \gamma(b,\mu)$ is a gamma distributed random variable independent of the beta distributed random variable $Y \sim \beta(2r, 1+b)$.

4 A multidimensional result

For $\nu > 0$, let us define $A_\infty^{(-\nu)} = \int_0^\infty e^{2B_s^{(-\nu)}}\,ds = \int_0^\infty e^{2(B_s - \nu s)}\,ds$. We are going to establish the following result, which is the first step towards a description of the joint distribution of $(A_\infty^{(-\nu)}, A_\infty^{(-(\nu+\alpha))})$ $(\nu > 0, \alpha > 0)$ (other multidimensional results can be found in Donati-Martin and Yor [11]; Donati-Martin, Ghomrasni, and Yor [10]; Sato and Yor [42]).

Proposition 4.1. *The conditional distribution, given $A_\infty^{(-\nu)} = a$, of $A_\infty^{(-(\nu+\alpha))}$ is the law of the random variable*

$$Z = a^2 \int_0^\infty e^{-2\alpha C_t} \frac{dt}{(a+t)^2},$$

where $C_t = \int_0^t \frac{ds}{X_s}$ and the law of X is $\mathbf{Q}_1^{2(1+\nu)}$, that is, X is a squared Bessel process, starting from 1, of dimension $\delta = 2(1+\nu)$.

Proof. The proof relies on two ingredients

- The first one is a special case of Lamperti's transformation: the process X defined by

$$e^{2B_t^{(\nu)}} = X_{A_t^{(\nu)}}$$

is a squared Bessel process, starting from 1, of dimension $\delta = 2(1+\nu)$.

- The second is the generalized Dufresne [13, 26, 27, 28] identity in law

$$\left(\frac{1}{A_t^{(-\nu)}}\,;t\geq 0\right)\stackrel{d}{=}\left(\frac{1}{A_t^{(\nu)}}+\frac{1}{\tilde{A}_\infty^{(-\nu)}}\,;t\geq 0\right),$$

where on the right-hand side the random variable $\tilde{A}_\infty^{(-\nu)}$ is independent from the process $(A_t^{(\nu)}\,;\,t\geq 0)$ and is distributed as $A_\infty^{(-\nu)}$.

The first step is a simple integration by parts,

$$A_t^{(-(\nu+\alpha))}=\int_0^t e^{-2\alpha s}\,dA_s^{(-\nu)}=e^{-2\alpha t}A_t^{(-\nu)}+2\alpha\int_0^t e^{-2\alpha s}A_s^{(-\nu)}\,ds.$$

Hence,

$$A_\infty^{(-(\nu+\alpha))}=2\alpha\int_0^\infty e^{-2\alpha s}A_s^{(-\nu)}\,ds.$$

Injecting the generalized Dufresne identity yields

$$\left(\frac{1}{A_t^{(-\nu)}}\,;t\geq 0\,;\,A_\infty^{(-(\nu+\alpha))}\right)\stackrel{d}{=}\left(\frac{1}{A_t^{(\nu)}}+\frac{1}{\tilde{A}_\infty^{(-\nu)}}\,;t\geq 0\,;\,2\alpha\int_0^\infty e^{-2\alpha s}\frac{A_s^{(\nu)}\tilde{A}_\infty^{(-\nu)}}{A_s^{(\nu)}+\tilde{A}_\infty^{(-\nu)}}\,ds\right)$$

and thus

$$(A_\infty^{(-\nu)},A_\infty^{(-(\nu+\alpha))})\stackrel{d}{=}\left(\tilde{A}_\infty^{(-\nu)},2\alpha\int_0^\infty e^{-2\alpha s}\frac{A_s^{(\nu)}\tilde{A}_\infty^{(-\nu)}}{A_s^{(\nu)}+\tilde{A}_\infty^{(-\nu)}}\,ds\right).$$

Therefore, the conditional distribution of $A_\infty^{(-(\nu+\alpha))}$ given $A_\infty^{(-\nu)}=a$ is the distribution of the random variable

$$\begin{aligned}Z&=2\alpha\int_0^\infty e^{-2\alpha s}\frac{aA_s^{(\nu)}}{a+A_s^{(\nu)}}\,ds\\&=2\alpha\int_0^\infty e^{-2\alpha s}\frac{aA_s^{(\nu)}}{a+A_s^{(\nu)}}e^{-2B_s^{(\nu)}}\,dA_s^{(\nu)}.\end{aligned}$$

The time change $C_t=\int_0^t\frac{du}{X_u}$ is the continuous inverse of the increasing process $A^{(\nu)}$, hence we have

$$Z=2\alpha\int_0^\infty e^{-2\alpha C_t}\frac{at}{a+t}\frac{dt}{X_t}=\int_0^\infty\frac{at}{a+t}\,d(e^{-2\alpha C_t})=\int_0^\infty e^{-2\alpha C_t}\frac{a^2\,dt}{(a+t)^2}.$$

□

Remark 4.1. We could have used, instead of Lamperti's transformation and Dufresne's identity, a combination of time reversal and time inversion arguments. This would result in a proof more in the spirit of Matsumoto and Yor [27].

Corollary 4.1. *The first moment of this conditional distribution is*

$$\mathbb{E}\left[A_\infty^{(-(\nu+\alpha))}\mid A_\infty^{(-\nu)}=a\right]=\int_{0<t,x<+\infty}dx\,dt\,\frac{a^2}{(a+t)^2}\frac{x^{\nu-\bar{\nu}/2}}{2t}e^{-\frac{x+1}{2t}}I_{\bar{\nu}}\left(\frac{\sqrt{x}}{t}\right),$$

where $\bar{\nu}=\sqrt{\nu^2+4\alpha}$.

Proof. From the absolute continuity relationship between squares of Bessel processes (see, e.g., Revuz and Yor [39, Chapter XI])

$$\mathbf{Q}_1^{2(1+\nu)}{}_{|\mathcal{F}_t} = (X_t)^{\nu/2} \exp\left(-\frac{\nu^2}{2} C_t\right) \mathbf{Q}^2_{1|\mathcal{F}_t},$$

we infer that

$$\mathbf{Q}_1^{2(1+\nu)}\left(e^{-2\alpha C_t}\right) = \mathbf{Q}_1^2\left(X_t^{\nu/2} \exp\left(-\left(2\alpha + \frac{1}{2}\nu^2\right) C_t\right)\right) = \mathbf{Q}_1^{2(1+\tilde{\nu})}\left(X_t^{\frac{\nu-\tilde{\nu}}{2}}\right).$$

Taking into account the explicit density of the square of a Bessel process $\mathbf{Q}_1^{2(1+\nu)}(X_t \in dx) = p_t^\nu(x)\, dx$ with

$$p_t^\nu(x) = \frac{x^{\nu/2}}{2t} e^{-\frac{x+1}{2t}} I_\nu(\sqrt{x}/t)\, 1_{(x>0)},$$

we finally get

$$\begin{aligned}
\mathbb{E}\left[A_\infty^{(-(\nu+\alpha))} \mid A_\infty^{(-\nu)} = a\right] &= \int_0^\infty \mathbb{E}\left[e^{-2\alpha C_t}\right] \frac{a^2\, dt}{(a+t)^2} \\
&= \int_0^\infty \mathbf{Q}_1^{2(1+\tilde{\nu})}\left(X_t^{\frac{\nu-\tilde{\nu}}{2}}\right) \frac{a^2\, dt}{(a+t)^2} \\
&= \int_{0<t,x<+\infty} dx\, dt\, \frac{a^2}{(a+t)^2} \frac{x^{(\nu-\tilde{\nu})/2}}{2t} p_t^{\tilde{\nu}}(x).
\end{aligned}$$

□

Remark 4.2. The generalized Ornstein–Uhlenbeck method seems to be quite well adapted to solve this problem: the law of $(A_\infty^{(-\nu-\alpha)}, A_\infty^{(-\nu)})$ is the unique invariant probability measure of the process $(e^{2(B_t-(\nu+\alpha)t)} \int_0^\infty e^{-2(B_s-(\nu+\alpha)s)} ds,\ e^{2(B_t-\nu t)} \int_0^\infty e^{-2(B_s-\nu s)} ds)$. Unfortunately, until now, we have not been able to solve the very simple partial differential equation satisfied by the density.

The problem of finding the joint law of $(A_\infty^{(-\nu)}, A_\infty^{(-(\nu+\alpha))})$ is closely related with the one studied by Rubin and Song [41], who study the joint law of $(A_\infty^{(-\nu)}, Y_\infty^{(-\nu)})$, with $Y_\infty^{(-\nu)} = \int_0^\infty s\, e^{2B_s^{(-\nu)}}\, ds$. The above results imply that

$$\left(A_\infty^{(-\nu)}, \frac{A_\infty^{(-\nu)} - A_\infty^{(-(\nu+\alpha))}}{\alpha}\right) \stackrel{d}{=} \left(\tilde{A}_\infty^{(-\nu)}, 2\int_0^\infty e^{-2\alpha s} \frac{(\tilde{A}_\infty^{(-\nu)})^2}{A_s^{(\nu)} + \tilde{A}_\infty^{(-\nu)}}\, ds\right),$$

so that, letting $\alpha \to 0$, we obtain:

$$(A_\infty^{(-\nu)}, Y_\infty^{(-\nu)}) \stackrel{d}{=} \left(\tilde{A}_\infty^{(-\nu)}, \int_0^\infty \frac{(\tilde{A}_\infty^{(-\nu)})^2}{A_s^{(\nu)} + \tilde{A}_\infty^{(-\nu)}}\, ds\right).$$

In particular, using classical computations involving the Whittaker functions and the explicit form of the densities $p_t^\nu(x)$ of the Bessel process, we recover Lemma 3 of [41] from the above identity in law.

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REFERENCES

[1] L. Arnold, *Random Dynamical Systems*, Monographs in Math., Springer, 1998.

[2] P. Baldi, E. Casadio-Tarabusi, A. Figa-Talamanca, and M. Yor, Non-symmetric hitting distributions on the hyperbolic half plane and subordinated perpetuities, *Rev. Mat. Ibero-Amer.*, to appear, 2001.

[3] J. P. Bouchaud and D. Sornette, The Black-Scholes option pricing problem in mathematical finance: Generalization and extensions for a large class of stochastic processes, *J. Phys.* I *France*, **4** (1994), 863–881.

[4] Ph. Carmona, The mean velocity of a Brownian motion in a random Lévy potential, *Ann. Probab.*, **25**-4 (1997), 1774–1788.

[5] Ph. Carmona, F. Petit, and M. Yor, Sur les fonctionnelles exponentielles de certains processus de Lévy, *Stoch. Stoch. Rep.*, **47**-1/2 (1994), 71–101 (in French); in M. Yor, *Exponential functionals of Brownian Motion and Related Processes*, Springer, 2001 (in English).

[6] Ph. Carmona, F. Petit, and M. Yor, On the distribution and asymptotic results for exponential functionals of Lévy processes, in *Exponential Functionals and Principal Values Related to Brownian Motion*, Biblioteca de la Revista Matematica Ibero-Americana, 1997, 73–126.

[7] Ph. Carmona, F. Petit, and M. Yor, A study of the Lamperti relation when the underlying semi-stable Markov process is a radial Cauchy process, in preparation, June 2000.

[8] A. Comtet and C. Monthus, On the flux distribution in a one dimensional disordered system, *J. Phys.* I *France*, **4** (1994), 635–653.

[9] A. Comtet, C. Monthus, and M. Yor, Exponential functionals of Brownian motion and disordered systems, *J. Appl. Probab.*, **2** (1998), 255–271.

[10] C. Donati-Martin, R. Ghomrasni, and M. Yor, On certain Markov processes attached to exponential functionals of Brownian motion: Application to Asian options; *Rev. Mat. Ibero-Amer.*, to appear, 2001.

[11] C. Donati-Martin and M. Yor, Some measure valued Markov processes attached to occupation times of Brownian motion, *Bernoulli*, **6**-1 (2000), 63–72.

[12] D. Dufresne, The distribution of a perpetuity, with application to risk theory and pension funding, *Scand. Actuar. J.*, 1990, 39–79.

[13] D. Dufresne, An affine property of the reciprocal Asian option process, Technical Report 63, Center for Actuarial studies, University of Melbourne, June 1998; *Osaka Math. J.*, to appear, 2001.

[14] H. Geman and M. Yor, Bessel processes, Asian options and perpetuities, *Math. Finance*, **3**-4 (1993), 349–375.

[15] V. Genon-Catalot, T. Jeantheau, and C. Laredo, Limit theorems for discretely observed stochastic volatility models, *Bernoulli*, **4**-3 (1998), 283–303.

[16] R. K. Getoor and M. J. Sharpe, Excursions of Brownian motion and Bessel processes, *Z. Wahrscheinlichkeitstheorie verw. Geb.*, **47** (1978), 83–106.

[17] H. Gjessing and J. Paulsen, Present value distributions with applications to ruin theory and stochastic equations, *Stoch. Proc. Appl.*, **71**-1 (1997), 123–144.

[18] H. Gjessing and J. Paulsen, Ruin theory with stochastic return on investments, *Adv. Appl. Probab.*, **29**-4 (1997), 965–985.

[19] J. Hammersley, On the statistical loss of long-period comets from the solar system II, in *Proc. 4th Berkeley Sympos. Math. Statist. and Probab.*, Vol. III, University of California Press, 1961, 17–78.

[20] K. Kawazu and H. Tanaka, On the maximum of diffusion process in a drifted Brownian environment, in *Séminaire de Probabilités* XXVII, Lecture Notes in Math. 1557, Springer, 1991, 78–85.

[21] K. Kawazu and H. Tanaka, A diffusion process in a Brownian environment with drift, *J. Math. Soc. Japan*, **49** (1997), 189–211.

[22] J. Kent, Some probabilistic properties of Bessel functions, *Ann. Probab.*, **6**-5 (1978), 760–770.

[23] F. B. Knight, On the sojourn times of killed Brownian motion, in *Séminaire de Probabilités* XII, Lecture Notes in Math. 649, Springer, 1978, 428–445.

[24] H. Kunita, On backward stochastic differential equations, *Stochastics*, **6** (1982), 293–313.

[25] J. Lamperti, Semi-stable Markov processes, *Z. Wahrscheinlichkeitstheorie verw. Geb.*, **22** (1972), 205–225.

[26] H. Matsumoto and M. Yor, On Bougerol and Dufresne's identities for exponential Brownian functionals, *Proc. Japan Acad. Ser.* A, **74**-10 (1998), 152–155.

[27] H. Matsumoto and M. Yor, A relationship between Brownian motions with opposite drifts via certain enlargements of the Brownian filtration, *Osaka Math. J.*, to appear, 2001.

[28] H. Matsumoto and M. Yor, A version of Pitman's $2M - X$ theorem for geometric Brownian motions, *C. R. Acad. Sci. Paris Ser.* I, **328**-11 (1999), 1067–1074.

[29] C. Monthus, *Etude de quelques fonctionnelles du mouvement brownien et de certaines propriétés de la diffusion unidimensionnelle en milieu aléatoire*, Ph.D. thesis, Université Paris VI, January 1995.

[30] T. Nilsen and J. Paulsen, On the distribution of a randomly discounted compound Poisson process, *Stoch. Proc. Appl.*, **61** (1996), 305–310.

[31] R. Norberg, Ruin problems with assets and liabilities of diffusion type, *Stoch. Proc. Appl.*, **81**-2 (1999), 255–269.

[32] D. Nualart, *The Malliavin Calculus and Related Topics*, Probability and Its Applications, Springer, 1995.

[33] E. Pardoux and P. Protter, Two-sided stochastic integral and calculus, *Probab. Theory Rel. Fields*, **76** (1987), 15–50.

[34] J. Paulsen, Risk theory in a stochastic economic environment, *Stoch. Proc. Appl.*, **46** (1993), 327–361.

[35] J. Paulsen and A. Hove, Markov chain Monte Carlo simulation of the distribution of some perpetuities, *Adv. Appl. Probab.*, **31**-1 (1999), 112–134.

[36] J. Pitman and M. Yor, Bessel processes and infinitely divisible laws, in D. Williams, ed., *Stochastic Integrals*, Lecture Notes in Math. 851, Springer, 1981.

[37] J. Pitman and M. Yor, Level crossings of a Cauchy process, *Ann. Probab.*, **11**-3 (1986), 780–792.

[38] M. Pollak and D. Siegmund, A diffusion process and its applications to detecting a change in the drift of Brownian motion, *Biometrika*, **72** (1985), 267–280.

[39] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer, 1991.

[40] J. Rosen and M. Yor, Tanaka formulae and renormalization for triple intersection of Brownian motion in the plane, *Ann. Probab.*, **19**-1 (1991), 142–159.

[41] H. Rubin and K. S. Song, Exact computation of the asymptotic efficiency of maximum likelihood estimators of a discontinuous signal in a gaussian white noise, *Ann. Statist.*, **23**-3 (1995), 732–739.

[42] S. Sato and M. Yor, Computations of moments for discounted brownian additive functionals, *J. Math. Kyoto Univ.*, **38**-3 (1998), 475–486.

[43] M. Yor, On some exponential functionals of Brownian motion, *Adv. Appl. Probab.*, **24** (1992), 509–531.

[44] M. Yor, Sur certaines fonctionnelles exponentielles du mouvement brownien réel, *J. Appl. Probab.*, **29** (1992), 202–208; in M. Yor, *Exponential Functionals of Brownian Motion and Related Processes*, Springer, 2001 (English translation).

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