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The Critical Temperature of a Directed Polymer in a Random Environment

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Abstract. In this paper, we find a necessary condition that ensures that the critical temperature of a directed polymer in a random environment is different from its lower bound obtained with the second moment method. Then we apply this criterion to the network $\mathbb{Z}^d$ and different distributions of the environment.

KEYWORDS: directed polymer, random environment, partition function

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1. Introduction

A large number of the disordered systems which have attracted the attention of mathematicians and physicists enjoy the following property. There exists a critical inverse temperature $\beta^{(c)}$ such that for $\beta < \beta^{(c)}$ (resp. $\beta > \beta^{(c)}$) the annealed and quenched free energies are equal (resp. different).

Usually, a second moment method yields a lower bound $\beta_2 \leq \beta^{(c)}$. Whether the equality $\beta_2 = \beta^{(c)}$ holds is an important issue which has received different answers. For example, there is equality for the Sherrington–Kirkpatrick model of spin glasses with no external field, whereas there is no equality for the corresponding mean field model, the REM [12].

For directed polymers in a random environment, we know that in general $\beta_2 < \beta^{(c)}$ for the mean field model of the tree [3,9], and the purpose of this paper is to answer this question on $\mathbb{Z}^d$ with a criterion depending on the dimension d and on the distribution of the environment.

Let $\mathbf{P}$ be the distribution of simple random walk $(\omega_n)_{n\in\mathbb{N}}$ on $\mathbb{Z}^d$, starting from the origin. The restriction of $\mathbf{P}$ to the set of nearest neighbor paths of length n

$$\Omega_n = \{\omega \in (\mathbb{Z}^d)^{n+1} : \omega_0 = 0, \|\omega_i - \omega_{i-1}\| = 1, 1 \le i \le n\}$$

is the uniform measure.

Given a random environment $(g(i,x))_{i\in\mathbb{N},x\in\mathbb{Z}^d}$, a set of i.i.d. random variables under the probability $\mathbf{Q}$, having finite exponential moments

$$\lambda(\beta) = \ln \mathbf{Q}\big(e^{\beta g(1,1)}\big) < +\infty \quad (\beta \in \mathbb{R}),$$

we define the energy of a path of length n as $H_n(\omega) = H_n(\omega, g) = \sum_{i=1}^n g(i, \omega_i)$ and the polymer measure

$$\mu_n(\omega) = \frac{1}{Z_n} \exp(\beta H_n(\omega)).$$

Hence the partition function is

$$Z_n = Z_n(\beta, g) = \mathbf{P}\big(\exp(\beta H_n)\big) = \frac{1}{(2d)^n} \sum_{\omega\in\Omega_n} \exp(\beta H_n(\omega)).$$

As usual, the behavior of a typical path under the random measure μ_n is dictated by the asymptotic behavior of the partition function.

Bolthausen [2] showed that $W_n = Z_n(\beta)\exp(-n\lambda(\beta))$ is a positive martingale, that converges almost surely to a finite random variable W_∞ that satisfies a $0-1$ law: $\mathbf{Q}(W_\infty = 0) \in \{0, 1\}$.

By a clever use of the FKG inequality, Comets and Yoshida [6] proved the existence of a critical temperature $\beta^{(c)}$ such that :

- for $0 \le \beta < \beta^{(c)}$, $W_\infty > 0$, a.s. (*weak disorder phase*);
- for $\beta > \beta_c$, $W_\infty = 0$, a.s. (*strong disorder phase*).

Furthermore, they established a diffusive behavior in weak disorder, and Carmona and Hu [5] proved a non-diffusive behavior in strong disorder.

For dimensions $d = 1, 2$ one can prove that $\beta^{(c)} = 0$ (see [4,7]), therefore we shall restrict ourselves, in the following, to dimensions $d \ge 3$. Let us observe that it is believed (see [5]) that this critical temperature coincides with the annealed/quenched transition critical temperature $\beta^{(c*)}$ which can be defined as

$$\beta^{(c*)} = \sup\{\beta > 0 : p(\beta) = \lambda(\beta)\}$$

with $p(\beta)$ the (limit) free energy

$$p(\beta) = \lim_{n\to+\infty} \frac{1}{n}\mathbf{Q}(\ln Z_n(\beta)) = \text{ a.s. } \lim_{n\to+\infty} \frac{1}{n} \ln Z_n(\beta).$$

To state our main result, we introduce $p(t,x)$, the probability that two independent random walks starting from 0 meet for the first time t at level x:

$$p(t,x) = \mathbf{P}^{\otimes 2}\big(\omega_j^1 \neq \omega_j^2, 1 \le j < t, \omega_t^1 = \omega_t^2 = x\big) \quad (t \ge 1, x \in \mathbb{Z}^d).$$

Let

$$\rho(\alpha) = \sum_{t,x} p(t,x)^{\alpha/2}, \quad \mathcal{D}_\rho = \{\alpha > 0, \rho(\alpha) < +\infty\},$$

$$h_\nu(\alpha) = -\sum_{t,x} \Big(\frac{p(t,x)^{\alpha/2}}{\rho(\alpha)}\Big) \ln \Big(\frac{p(t,x)^{\alpha/2}}{\rho(\alpha)}\Big) = \ln \rho(\alpha) - \alpha \frac{\rho'(\alpha)}{\rho(\alpha)} \quad (\alpha \in \mathcal{D}_\rho).$$

To avoid trivialities we shall assume that for $2 - \varepsilon < \alpha \le 2$ we have

$$\beta_\alpha = \sup\big\{\beta > 0 : \lambda(\alpha\beta) - \alpha\lambda(\beta) < -\ln \rho(\alpha)\big\} < +\infty,$$

and we consider another entropy

$$h_{\mathbf{Q}}(\alpha) = \mathbf{Q}\Big(\Big(\frac{\exp(\alpha\beta_\alpha g)}{\mathbf{Q}(\exp(\alpha\beta_\alpha g))}\Big) \ln \Big(\frac{\exp(\alpha\beta_\alpha g)}{\mathbf{Q}(\exp(\alpha\beta_\alpha g))}\Big)\Big) = \alpha\beta_\alpha\lambda'(\alpha\beta_\alpha) - \lambda(\alpha\beta_\alpha).$$

Theorem 1.1. *If $h_\nu(2) < h_{\mathbf{Q}}(2)$, then $\beta_2 < \beta^{(c)}$.*

Although this criterion is based on Theorem 2.1 by Derrida and Evans (see Section 2), it is much simpler to use (numerically). We only need to compute one number $h_\nu(2)$ for each graph $\mathbb{Z}^d$, instead of having to determine the whole function $\alpha \mapsto \rho(\alpha)$.

Let us stress the fact that the criterion of Theorem 1.1 does not compare to the criterion of Birkner [1] (which relies on an unpublished paper).

The structure of the paper is the following: Section 2 contains Derrida and Evans Theorem 2.1, Section 3 — the proof of Theorem 1.1. In Section 4 some numerical applications to different distributions of the environment are developed. These numerical simulations are done to overcome the theoretical difficulty of understanding the network entropy $h_\nu(2)$. This quantity is approximated by a finite sum whose terms are simulated using Monte-Carlo method. A rough upper bound is established to estimate the error. The appendix contains the computer programs we used.

2. The fractional moment method

We shall give a self contained proof of the following result by Derrida and Evans [8].

Theorem 2.1. *If there exists $1 < \alpha \le 2$ such that $\lambda(\alpha\beta) - \alpha\lambda(\beta) < -\ln \rho(\alpha)$, then $\beta \le \beta^{(c)}$.*

For $\alpha = 2$, we have $\rho(2) = \mathbf{P}^{\otimes 2}(\exists t \geq 1, \omega_t^1 = \omega_t^2)$ and this is the second moment criteria (see [2]).

Proof. The first step of the proof is the use of the following characterization of the weak disorder phase (see [4, 7]):

$$W_\infty > 0 \text{ a.s.} \iff (W_n)_{n\in\mathbb{N}} \text{ is uniformly integrable.}$$

Hence, if $\sup_n \mathbf{Q}(W_n(\beta)^\alpha) < +\infty$ for some $\alpha > 1$ then we are in weak disorder and $\beta \leq \beta^{(c)}$. In order to obtain some improvement on the second moment method, we shall restrict ourselves to $\alpha \in]1, 2]$ and use the inequality (for $\gamma = \alpha/2$):

$$\Big(\sum x_i\Big)^\gamma \leq \sum x_i^\gamma, \quad \gamma \in [0, 1],\ x_i \geq 0. \tag{2.1}$$

We first compute the second moment of Z_n. To do this we introduce two independent random walks and then split the expectations according to their meeting times: if $r = (t_i, x_i, 1 \leq i \leq m) \in (\mathbb{N}_n \times \mathbb{Z}_n^d)^m$ we consider the event

$$\big\{\omega^1 \overset{r}{=} \omega^2\big\} = \big\{\omega_{t_i}^1 = \omega_{t_i}^2 = x_i,\ 1 \leq i \leq m,\ \omega_t^1 \neq \omega_t^2, t \notin \{t_i\}\big\},$$

and compute

$$Z_n^2 = \mathbf{P}^{\otimes 2}\big(\exp(\beta(H_n(\omega^1) + H_n(\omega^2)))\big) = \sum_{m=0}^n \sum_{r\in(\mathbb{N}_n\times\mathbb{Z}_n^d)^m} Y(r),$$

with $Y(r) = \mathbf{P}^{\otimes 2}\big(\exp(\beta(H_n(\omega^1) + H_n(\omega^2)))\mathbf{1}\{\omega^1 \overset{r}{=} \omega^2\}\big)$.

Combining with inequality (2.1), we obtain,

$$\mathbf{Q}(Z_n^\alpha) = \mathbf{Q}\big((Z_n^2)^{\alpha/2}\big) = \mathbf{Q}\bigg(\Big\{\sum_{m=0}^n \sum_r Y(r)\Big\}^{\alpha/2}\bigg) \leq \sum_{m=0}^n \sum_r \mathbf{Q}\big[Y(r)^{\alpha/2}\big].$$

Let's concentrate now on the quantity $Y(r)$. We define the partial Hamiltonian:

$$H_{j_1}^{j_2}(\omega) = \sum_{i=j_1+1}^{j_2} g(i, \omega_i).$$

We can thus decompose, noting $\omega_{i,j} = (\omega_k)_{k\in\{t_i,\ldots,t_j\}}$, $t_0 = 0$ and $t_{m+1} = n$,

$$\begin{aligned} Y(r) = \mathbf{P}\bigg\{ &\prod_{i=1}^m \exp\big\{\beta(H_{t_{i-1}}^{t_i}(\omega^1) + H_{t_{i-1}}^{t_i}(\omega^2))\big\}\mathbf{1}\{\omega_{i-1,i}^1 \overset{(t_i,x_i)}{=} \omega_{i-1,i}^2\} \\ &\times \exp\big\{\beta(H_{t_m}^n(\omega^1) + H_{t_m}^n(\omega^2))\big\}\mathbf{1}\{\omega_{m,m+1}^1 \neq \omega_{m,m+1}^2\}\bigg\} \\ = &\prod_{i=1}^m Y_{i-1,i} \times \widetilde{Y}_{m,n}, \end{aligned}$$

with

$$\begin{aligned} Y_{i-1,i} = & \mathbf{P}^{\otimes 2}\big(\exp\{\beta(H^{t_i}_{t_{i-1}}(\omega^1)+H^{t_i}_{t_{i-1}}(\omega^2))\} \\ & \times \mathbf{1}\{\omega^1_{t_j}=\omega^2_{t_j}=x_j,\, j=i-1,i;\omega^1_t\neq\omega^2_t, t_{i-1}<t<t_i\}\big), \\ \widetilde{Y}_{m,n} = & \mathbf{P}^{\otimes 2}\big(\exp\{\beta(H^{n}_{t_m}(\omega^1)+H^{n}_{t_m}(\omega^2))\} \\ & \times \mathbf{1}\{\omega^1_{t_m}=\omega^2_{t_m}=x_m;\omega^1_t\neq\omega^2_t, t_m<t\leq n\}\big). \end{aligned}$$

Hence, using the independence of the environment with respect to the temporal evolution:

$$\mathbf{Q}[Z^\alpha_n] \leq \sum_{m=0}^{n}\sum_{r}\prod_{i=1}^{m}\mathbf{Q}\big[Y^{\alpha/2}_{i-1,i}\big]\mathbf{Q}\big[\widetilde{Y}^{\alpha/2}_{m,n}\big].$$

Since the random walks never meet after time t_m, Jensen's inequality yields

$$\begin{aligned} \mathbf{Q}\big[\widetilde{Y}^{\alpha/2}_{m,n}\big] & \leq \mathbf{P}\big(\omega^1_j\neq\omega^2_j, t_m<j\leq n;\omega^1_{t_m}=\omega^2_{t_m}=x_m\big)^{\alpha/2}\exp(\alpha(n-t_m)\lambda(\beta)) \\ & \leq \exp(-\alpha t_m\lambda(\beta)). \end{aligned}$$

The environment is equally distributed, so we can write our upper bound

$$\begin{aligned} \limsup_n \mathbf{Q}[W^\alpha_n] & \leq \sum_{m=0}^{\infty}\sum_{r\in(\mathbb{N}\times\mathbb{Z}^d)^m}\prod_{i=1}^{m}\mathbf{Q}\big[\exp\big(-\alpha(t_i-t_{i-1})\lambda(\beta)\big)Y^{\alpha/2}_{i-1,i}\big] \\ & \leq \sum_{m=0}^{\infty}\Big\{\sum_{t_1\in\mathbb{N},x_1\in\mathbb{Z}^d}\mathbf{Q}\big[\exp\big(-\alpha t_1\lambda(\beta)\big)Y^{\alpha/2}_{0,1}\big]\Big\}^m. \end{aligned}$$

Thanks again to the independence of the environment, denoting $(t,x)=(t_1,x_1)$ and using Jensen's inequality, we get

$$\begin{aligned} & \mathbf{Q}\Big[\frac{Y^{\alpha/2}_{0,1}}{\exp(\alpha t\lambda(\beta))}\Big] \\ & = \exp(-\alpha t\lambda(\beta)) \\ & \times\mathbf{Q}\left[\mathbf{P}\left[\exp\left(\beta\sum_{i=1}^{t-1}g(i,x^1_i)+g(i,x^2_i)\right)\exp(2\beta g(t,x))\mathbf{1}\{\omega^1\overset{(t,x)}{=}\omega^2\}\right]^{\alpha/2}\right] \\ & = \exp(-\alpha t\lambda(\beta))\exp(\lambda(\alpha\beta)) \\ & \times\mathbf{Q}\left[\mathbf{P}\left[\exp\left(\beta\sum_{i=1}^{t-1}g(i,x^1_i)+g(i,x^2_i)\right)\mathbf{1}\{\omega^1\overset{(t,x)}{=}\omega^2\}\right]^{\alpha/2}\right] \\ & \leq \exp(\lambda(\alpha\beta)-\alpha t\lambda(\beta))\mathbf{QP}\left[\exp\left(\beta\sum_{i=1}^{t-1}g(i,x^1_i)+g(i,x^2_i)\right)\mathbf{1}\{\omega^1\overset{(t,x)}{=}\omega^2\}\right]^{\alpha/2} \end{aligned}$$

$$= \exp(\lambda(\alpha\beta) - \alpha t\lambda(\beta)) \exp\Big(2\frac{\alpha}{2}\lambda(\beta)(t-1)\Big) \mathbf{P}\big(\omega^1 \overset{(t,x)}{=} \omega^2\big)^{\alpha/2}$$
$$= \exp(\lambda(\alpha\beta) - \alpha\lambda(\beta)) p(t,x)^{\alpha/2}.$$

Finally, we find the following upper bound:

$$\limsup_n \mathbf{Q}\big[W_n^{\alpha/2}\big] \le \sum_{m=0}^{\infty} \big\{e^{\lambda(\alpha\beta)-\alpha\lambda(\beta)}\rho(\alpha)\big\}^m.$$

Therefore, if there exists $\alpha \in (1,2]$ such that

$$\lambda(\alpha\beta) - \alpha\lambda(\beta) < -\ln\rho(\alpha),$$

then the martingale $(W_n(\beta))_n$ is uniformly integrable. □

Remark 2.1. For a directed polymer on a tree, this method yields the critical temperature by letting $\alpha \downarrow 1$.

3. Proof of Theorem 1.1

The first subsections are devoted to the proof of Theorem 1.1. Since we derive Theorem 1.1 from Theorem 2.1, we can wonder if we loose something in the process, but we shall see this is not the case for a Gaussian environment.

3.1. The function ρ

Since $0 \le p(t,x) \le 1$, the function ρ is non increasing on $(\alpha_0, +\infty)$ with $\alpha_0 = \inf\{\alpha > 0 : \rho(\alpha) < +\infty\}$. The properties of ρ we use in the sequel are summarized in the

Proposition 3.1.

(1) $\rho(2) = 1 - q_d = \mathbf{P}^{\otimes 2}(\exists t \ge 1, \omega_t^1 = \omega_t^2) < 1$ *(for* $d \ge 3$*).*

(2) $4/d \le \alpha_0 \le 1 + 2/d < 2$.

(3) *There exists* $1 < \alpha_1 < 2$ *such that* $\rho(\alpha_1) = 1$.

Proof. (3) is an easy consequence of (2), (1) and the continuity of ρ.

(1) We have:

$$\rho(2) = \sum_{t,x} p(t,x) = \sum_t \mathbf{P}^{\otimes 2}\Big(\mathbf{1}\{(\omega_j^1 \ne \omega_j^2, \forall j < t)\} \sum_x \mathbf{1}\{(\omega_t^1 = \omega_t^2 = x)\}\Big)$$
$$= \sum_t \mathbf{P}^{\otimes 2}\big(\omega_j^1 \ne \omega_j^2,\ \forall j < t; \omega_t^1 = \omega_t^2\big)$$
$$= \mathbf{P}^{\otimes 2}\big(\exists t \ge 1, \omega_t^1 = \omega_t^2\big).$$

The lower bound. For $1 \le \alpha \le 2$, we have by inequality (2.1),

$$\rho(\alpha) = \sum_{t\ge 1}\sum_x p(t,x)^{\alpha/2} \ge \sum_{t\ge 1}\Big(\sum_x p(t,x)\Big)^{\alpha/2}$$
$$= \sum_t \mathbf{P}^{\otimes 2}\big(\omega_t^1 = \omega_t^2, \omega_j^1 \ne \omega_j^2, \forall j < t\big)^{\alpha/2}.$$

Since Griffin [10] proved that

$$\mathbf{P}^{\otimes 2}\big(\omega_t^1 = \omega_t^2, \omega_j^1 \ne \omega_j^2, \forall j < t\big) \asymp \mathbf{P}^{\otimes 2}\big(\omega_t^1 = \omega_t^2\big) \asymp \mathbf{P}(\omega_{2t} = 0) \asymp t^{-d/2},$$

we see that $\rho(\alpha) = +\infty$ when $\alpha \le 4/d$.

The upper bound. If we suppress the avoiding condition in the definition of $p(t,x)$, we obtain

$$p(t,x) \le r(t,x)^2, \quad \text{with} \;\; r(t,x) = \mathbf{P}(\omega_t = x).$$

We are going to prove that $\sum r(t,x)^\alpha < +\infty$ if $\alpha > 1 + 2/d$ and this will imply that $\alpha_0 \le 1 + 2/d$.

First we apply the local central limit theorem (see [11, Theorem 1.2.1]):

$$\sup_x |r(t,x) - \bar r(t,x)| \le C t^{-1+d/2}$$

and thus we shall prove that $\bar\rho(\alpha) = \sum \bar r(t,x)^\alpha < +\infty$, if $\alpha > 1 + 2/d$, with

$$\bar r(t,x) = 2\Big(\frac{d}{2\pi t}\Big)^{d/2} \exp\Big(-\frac{d|x|^2}{2t}\Big).$$

Since

$$N(\sqrt{R}) = \sharp\big\{x \in \mathbb{Z}^d, \sqrt{R} \le x \le \sqrt{R}+1\big\} \sim C_d R^{d/2-1},$$

we need to show that

$$\sum_{t=1}^{\infty} \frac{1}{t^{\alpha d/2}} \sum_{R=1}^{t} R^{d/2-1} e^{-\alpha d R/2t} < +\infty.$$

Comparing series and integral for a monotone function, this amounts to check that

$$\sum_{R\ge 1} R^{d/2-1} \int_R^{+\infty} t^{-\alpha d/2} e^{-(\alpha d/2t)R}\, dt$$
$$= \Big(\int_1^{+\infty} u^{-\alpha d/2} e^{-\alpha d/2u}\, du\Big) \sum_{R\ge 1} R^{(d/2)(1-\alpha)} < +\infty$$

and this is satisfied since $\alpha > 1 + 2/d$. □

3.2. The proof of Theorem 1.1

Fix $\alpha \in (2-\varepsilon, 2]$, so that if

$$\Psi(\alpha,\beta) = \lambda(\alpha\beta) - \alpha\lambda(\beta) + \ln\rho(\alpha),$$

the function $\beta \to \Psi(\alpha,\beta)$ has the following properties:

- It is C^1 and $\partial\Psi/\partial\beta = \alpha(\lambda'(\alpha\beta) - \lambda'(\beta)) > 0$ for $\beta > 0$ since λ is strictly convex (for a non degenerate environment).
- $\Psi(\alpha, 0) = \ln\rho(\alpha) < 0$ if ε is small enough since $\rho(2) = 1 - q_d < 1$.
- There exists $\beta > 0$ such that $\Psi(\alpha,\beta) > 0$ by assumption.

Therefore, there is a unique $\beta_\alpha > 0$ such that $\Psi(\alpha,\beta_\alpha) = 0$.

If we prove that $(\partial\beta_\alpha/\partial\alpha)\big|_{\alpha=2} < 0$ then we are done, since there exists then $\alpha \in (1,2)$ such that $\beta_\alpha > \beta_2$, and thus, by definition of β_α, there exists $\gamma \in (\beta_2, \beta_\alpha)$ such that $\Psi(\alpha,\gamma) < 0$ and we apply Theorem 2.1.

By the implicit function theorem, we have

$$\frac{\partial\beta_\alpha}{\partial\alpha} = -\frac{\partial\Psi/\partial\alpha}{\partial\Psi/\partial\beta}(\alpha,\beta_\alpha)$$

and thanks to $\partial\Psi/\partial\beta = \alpha(\lambda'(\alpha\beta) - \lambda'(\beta)) > 0$ we only need to prove that $\partial\Psi/\partial\alpha > 0$.

A straightforward computation yields

$$\alpha\,\frac{\partial\Psi}{\partial\alpha}(\alpha,\beta_\alpha) = h_{\mathbf{Q}}(\alpha) - h_\nu(\alpha),$$

therefore if $h_{\mathbf{Q}}(2) - h_\nu(2) > 0$, then $\beta \le \beta^{(c)}$.

3.3. In a Gaussian environment

In Gaussian environment, we prove that function $\alpha \mapsto \lg(\beta_\alpha)$ is concave. Thus, $h_{\mathbf{Q}}(2) > h_\nu(2)$ is equivalent with the existence of $\alpha^\star$ such that $\beta_2 < \beta_{\alpha^\star} < \beta^{(c)}$.

Indeed, we have $\lambda(\beta) = \beta^2/2$. Therefore, for any $\alpha \ge \alpha_1$ (that is $\rho(\alpha) \le 1$), we have

$$\beta_\alpha = 2\Big(-\frac{\ln\rho(\alpha)}{\alpha(\alpha-1)}\Big)^{1/2}.$$

Tedious but straightforward computations yield successively

$$\partial_\alpha\Big(\frac{\beta_\alpha^2}{2}\Big) = \frac{\Gamma(\alpha)}{\alpha(\alpha-1)}, \quad \text{with} \quad \Gamma(\alpha) = (2\alpha-1)\ln\rho(\alpha) - (\alpha-1)\frac{\rho'(\alpha)}{\rho(\alpha)},$$
$$\partial_\alpha\Gamma(\alpha) = 2\ln\rho(\alpha) - \alpha(\alpha-1)\operatorname{Var}_{\nu_\alpha}\big(\ln p(\cdot,\cdot)^{1/2}\big) < 0$$

with ν_α the probability measure defined on $\mathbb{N}^* \times \mathbb{Z}^d$ by $\nu(t,x) = p(t,x)^{\alpha/2}/\rho(\alpha)$.

Therefore, the function $\alpha \to \lambda(\beta_\alpha)$ is concave, and the assumptions $h_{\mathbf{Q}}(2) - h_\nu(2) > 0$, that is $\partial_\alpha \beta_\alpha \big|_{\alpha=2} < 0$, and there exist $\alpha < 2, \beta_\alpha > \beta_2$ are equivalent, so Theorem 1.1 is not weaker than Theorem 2.1 in the Gaussian case.

4. Numerical results

In this section we present some applications of Theorem 1.1. Studying the function ρ, we have seen that the quantities $p(t,x)$ are very difficult to understand theoretically since they are related to self-avoiding random walks. Thus, to check the entropic condition $h_\nu(2) < h_{\mathbf{Q}}(2)$, we use numerical simulations. In the following lines we explain to what extent our numerical simulations provide an answer to the initial question: *Does* $\beta_2 = \beta_c$? We expect that the speed of convergence is faster than the one obtained in the following.

We compute approximate values of

$$h_\nu(2) = -\sum_{t\in\mathbb{N}} \sum_{x\in\mathbb{Z}^d} \frac{p(t,x)}{1-q_d} \ln \frac{p(t,x)}{1-q_d}.$$

We simulate $N \sim 50000$ random walks of length 1000 and look at their first meeting time to obtain an approximation of $p(t,x)$ for $t \le 1000$. The value of $\tilde{q}_d$ thus obtained is very close to the one (denoted q_d) obtained by Griffin [10].

d	$1-q_d$	$\tilde{q}_d - q_d$	$h_\nu(2)$
3	0.340	-0.005	4.808
4	0.193	-0.003	3.855
5	0.135	-0.001	3.608

Thus we get a close approximation of the environment entropy $h_\nu(2)$. Indeed, let us recall that $\alpha_0 \le 1 + 2/d \le 1.8$. A rough approximation gives

$$\begin{aligned} -\sum_{t\ge t_0} \sum_{x\in\mathbb{Z}^d} \frac{p(t,x)}{1-q_d} \ln \frac{p(t,x)}{1-q_d} &\le C \sum_{t\ge t_0} \sum_{x\in\mathbb{Z}^d} \left(\frac{p(t,x)}{1-q_d}\right)^{0.9} \\ &\le C \sum_{t\ge t_0} \sum_{x;|x|\le t_0} r(t,x)^{1.8} \le C t_0^{-0.2}. \end{aligned}$$

Then we compute $h_{\mathbf{Q}}(2) = 2\beta_2 \lambda'(2\beta_2) - \lambda(2\beta_2)$ for different environments. We estimate the inverse temperature β_2 solving numerically the equation

$$\lambda(2\beta_2) - 2\lambda(\beta_2) = -\ln(1-q_d).$$

The three environments considered are symmetric. For the Binomial environment, $|g(1,1)|$ is binomial with parameters $n = 5$ and $p = 0.01$. For the

Poisson environment, $|g(1,1)|$ is Poisson of parameter $k = 0.0001$. In opposition to these environments, simple computation entails that Gaussian behavior does not depend on the variance.

d	$h_\nu(2)$	$h_{\mathbf{Q}}(2)$		
		Binomial	Poisson	Gaussian
3	4.808	4.14	6.418	2.158
4	3.855	6.228	10.295	3.29
5	3.608	7.421	12.726	4.004

We notice that for a binomial environment, our criteria ensures $\beta_2 \neq \beta_c$ even in dimension 3.

What is the influence of the dimension d? On the one hand, it is easy to prove that the function $d \mapsto h_{\mathbf{Q}}(2)$ is non decreasing. On the other hand, we expect the function $d \mapsto h_\nu(2)$ to be non increasing, but we are unable to prove it. Consequently, it seems that the critical dimension for our criteria is 4 for Binomial, 3 for Poisson and 5 for Gaussian environments.

Appendix: the program

We give here the matlab program used to compute $p(t,x)$. The result is a row vector. This vector doesn't contain the information of which t and which x are considered.

```
%P(d,n,N) 1<i<n
%d : dimension
%n : maximal length of the r.w.
%N : number of experiments

%Initialising the counter
%colomn : [coord of the meeting point; moment; counter]
R=zeros(1,d+1);

for i=1:N
    %Constructing the two r.w.
    x=2*floor(2*rand(1,2*n))-1;   %direction of the jump
    t=floor(d*rand(1,2*n))+1;     %jumping coord.

    %number of the jumping coordinate increments
    t=t+[0:d:d*(2*n-1)];

    %Increments matrix
    xi=zeros(2*d,n);
```

```
    xi(t)=x;

    X=cumsum(xi,2);

    clear xi;clear x;clear t;

    %row vector: 1 if the r.w. are at the same point, 0 otherwise
    z=prod(double(X(1:d,:)==X(d+1:2*d,:)));
    X=X(1:d,:);
    %X : [coord time]
    X=[(1:n)' X'];

    if z==zeros(1,length(z))
        ;
    else
        I=find(z');        %meeting times
        k=I(1);
        X=X(k,:);           %meeting points
        R=[R;X];
    end
    clear X; clear z;clear k;
end
%suppress the first row
[r1 r2]=size(R);
R=R(2:r1,:);
r1=r1-1;

%order the rows in alphabetic order
R=sortrows(R);
%1 if 2 successive rows are equal in R, 0 otherwise
z=prod(double(R==[zeros(1,d+1);R(1:r1-1,:)]),2);
I=find(z==zeros(length(z),1));
clear z;
R=R(I,:);
i=length(I);

%number of r.w. meeting in each site
c=[I(2:i);r1+1]-I;

y=c/N;
```

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